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Linear choosability of graphs

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Abstract

A proper vertex coloring of a non-oriented graph G is *linear* if the graph induced by the vertices of any two color classes is a forest of paths. A graph G is linearly L -list colorable if for a given list assignment $L = \{L(v) : v \in V(G)\}$, there exists a linear coloring c of G such that $c(v) \in L(v)$ for all $v \in V(G)$. If G is linearly L -list colorable for any list assignment with $|L(v)| \geq k$ for all $v \in V(G)$, then G is said to be linearly k -choosable. In this paper, we investigate the linear choosability for some families of graphs: graphs with small maximum degree, with given maximum average degree, outerplanar and planar graphs. Moreover, we prove that deciding whether a bipartite subcubic planar graph is linearly 3-colorable is an NP-complete problem.

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1. Introduction

The notion of acyclic colorings was introduced by Grünbaum [8] in 1973: a vertex coloring is said to be acyclic if it is proper (no two adjacent vertices have the same color), and if there is no bicolored cycle (the subgraph induced by the union of any two color classes is a forest).

A coloring c such that for every vertex $v \in V(G)$, no color appears more than $k - 1$ times in the neighborhood of v , is called a k -frugal coloring. The notion of k -frugality was introduced by Hind et al. in [9].

Yuster mixed these two notions in [16], while introducing the concept of linear coloring. A *linear coloring* of a non-oriented graph is an acyclic and 3-frugal coloring. It can also be seen as a coloring such that the subgraph induced by the union of any two color classes is a forest of paths (an acyclic graph with maximum degree at most two). The *linear chromatic number* of a graph G , denoted by $\Lambda(G)$, is the minimum number of colors in a linear coloring of G .

Yuster proved in [16] that $\Lambda(G) = O(\Delta(G)^{3/2})$ in the general case, and he constructed graphs for which $\Lambda(G) = \Omega(\Delta(G)^{3/2})$.

The concept of choosability was introduced by Vizing [15], Erdős et al. [6]. This generalization of the notion of coloring has been applied to various problems, and more particularly to the field of coloring under constraints (acyclic choosability [3], (a, b) -choosability [13], k -improper l -choosability [12]). In this paper, we investigate the linear choosability for some families of graphs.

A graph G is linearly L -colorable if for a given list assignment $L = \{L(v) : v \in V(G)\}$, there exists a linear coloring c of G such that $c(v) \in L(v)$ for each vertex v . Such a coloring is called a *linear L -coloring* of G . If G is linearly

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L -colorable for any assignment L verifying $\forall v \in V(G), |L(v)| \geq k$, then G is said to be linearly k -choosable. The smallest integer k such that the graph G is linearly k -choosable is called the *linear list-chromatic number*, denoted by $A^l(G)$.

We begin with some definitions and basic results (Section 2). Section 3 is dedicated to the study of graphs with small maximum degree: we prove that $A^l(G) \leq 5$ when $\Delta(G) \leq 3$, and $A^l(G) \leq 9$ when $\Delta(G) \leq 4$. In Section 4, we use a canonical decomposition to prove that every outerplanar graph G with maximum degree Δ verifies $A^l(G) \leq \lceil \Delta/2 \rceil + 2$. In Section 5, we give bounds for graphs with bounded maximum average degree. In Section 6, we prove that every planar graph of maximum degree $\Delta \geq 12$ verifies $A^l(G) \leq \Delta + 26$. Finally, we prove that determining whether a bipartite subcubic planar graph is linearly 3-colorable is an NP-complete problem (Section 7).

In the following, we recall some definitions and notations. Let G be a simple graph (i.e. without multiple edges or loops), $V(G)$ its set of vertices and $E(G)$ its set of edges. Let $N(v)$ be the neighborhood of the vertex $v \in V(G)$, i.e. the set of the vertices adjacent to v . The degree of a vertex v is denoted by $d(v)$, and the maximum degree of the graph G is denoted by $\Delta(G)$. A vertex with degree d (resp. at most d , at least d) is called a d -vertex (resp. $\leq d$ -vertex, $\geq d$ -vertex). A graph is said d -regular if it only contains d -vertices.

A 3-frugal coloring was defined above as a proper coloring of the vertices of a graph, such that no color appears more than twice in each neighborhood. In the following, we will also use a slight abuse of notation, by saying that the 3-frugality of a vertex v is *respected* or *preserved*, when no color appears more than twice in $N(v)$.

2. First results

A linear coloring is a 3-frugal coloring, so there are at least $\lceil d/2 \rceil$ distinct colors in the neighborhood of each d -vertex. Thus we have the following bound:

Proposition 1. *If G is a graph with maximum degree Δ , then $A(G) \geq \lceil \Delta/2 \rceil + 1$.*

As $A^l(G) \geq A(G)$ for every graph G , we also have $A^l(G) \geq \lceil \Delta/2 \rceil + 1$. Moreover, this bound is tight for some families of graphs, such as trees.

Proposition 2. *If G is a tree with maximum degree Δ , then $A^l(G) = \lceil \Delta/2 \rceil + 1$.*

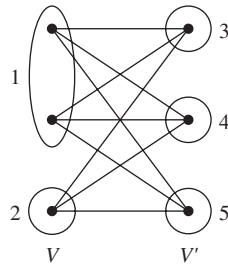
Proof. Let L be an assignment of color lists of size at least $\lceil \Delta/2 \rceil + 1$ to the vertices of G . We proceed by induction on the order of the graph. Let v be a leaf of G , and let u be v 's neighbor. By the induction assumption, there exists a linear L -coloring c of $G \setminus v$. We now extend c to v by finding a color $c(v) \in L(v)$ such that the coloring obtained is linear. We only forbid to v the color $c(u)$ and the colors appearing at least twice in u 's neighborhood. This is sufficient to obtain a proper and 3-frugal coloring, and thus a linear coloring of the tree G . There are at most $1 + \lceil \Delta/2 \rceil - 1 = \lceil \Delta/2 \rceil$ forbidden colors. Since $|L(v)| \geq \lceil \Delta/2 \rceil + 1$, it is possible to color v with a color from its list. \square

Let $K_{m,n}$ be the complete bipartite graph with stable sets V and V' of size m and n , respectively. We show the following result:

Proposition 3. *If $m \geq n$, $A^l(K_{m,n}) = A(K_{m,n}) = \lceil m/2 \rceil + n$.*

Proof. To prove that $A(K_{m,n}) \geq \lceil m/2 \rceil + n$, observe that if two vertices of a same set V or V' have the same color, then all the vertices of the other set must have distinct colors (otherwise there would be a bicolored cycle of length four). Moreover a given color cannot appear more than twice in $V \cup V'$ since otherwise the 3-frugality would not be respected. Hence, the best solution is to assign each color to a pair of vertices in the largest set, and to color all the remaining vertices with distinct colors (see Fig. 1).

We now prove that $A^l(K_{m,n}) \leq \lceil m/2 \rceil + n$, which completes the proof of Proposition 3. Let L be an assignment of lists of size at least $\lceil m/2 \rceil + n$ for the vertices of $K_{m,n}$. We first color the vertices v_i of V (the largest set): we only forbid to v_i the colors appearing already twice among the already colored vertices. We then color the vertices v'_i of V' : we forbid to each v'_i all the colors that have already been used. With lists of size at least $\lceil m/2 \rceil + n$, it is possible to color all the vertices, and the final coloring is linear. \square

Fig. 1. A linear coloring of $K_{3,3}$.

Observe that the linear (list-)chromatic number of $K_{n,n}$ is asymptotically equivalent to $3\Delta/2$.

A 2-degenerate graph G is a graph such that every subgraph of G contains a vertex of degree at most two. We prove the following proposition:

Proposition 4. *If G is a 2-degenerate graph of maximum degree Δ , then $\Lambda^l(G) \leq \Delta + 2$.*

Proof. Let G be a counterexample of minimum order. There exists an assignment L of lists (with $|L(v)| \geq \Delta + 2$ for all v) such that G is not linearly L -colorable. We show that G does not contain any ≤ 2 -vertex.

Let v be a vertex of degree one in G . The graph $G \setminus v$ is a proper subgraph of G , thus it is a 2-degenerate graph with order strictly less than that of G . By the minimality of G , there exists a linear L -coloring c of $G \setminus v$. By coloring the vertex v with a color from its list $L(v)$, we extend the coloring c to the whole graph G , thus obtaining a contradiction. We choose for v a color distinct from the color of its neighbor w and from the colors appearing twice in w 's neighborhood. At most $\lceil \Delta/2 \rceil - 1 + 1 = \lceil \Delta/2 \rceil$ colors are forbidden to v , so it is possible to color it with a color from its list $L(v)$, as $|L(v)| \geq \Delta + 2$.

We now prove that G does not contain any vertex of degree two. Let v be a vertex of degree two in G , with neighbors u and w . As previously, the graph $G \setminus v$ is 2-degenerate with order strictly less than G , so there exists a linear L -coloring c of $G \setminus v$. Let us extend the coloring c to the whole graph G , i.e. find a color $c(v) \in L(v)$ such that the coloring obtained is linear. We forbid to v the colors belonging to the set \mathcal{C} defined as follows. A color a is in \mathcal{C} if one of the following conditions is verified:

- one neighbor of u and one neighbor of w are both colored with a (a bicolored cycle could be created if v was also colored with a);
- two neighbors of u are colored with a (the 3-frugality of u would not be preserved if v was also colored with a);
- two neighbors of w are colored with a (3-frugality of w).

Observe that $|\mathcal{C}| \leq \Delta - 1$, since any color of \mathcal{C} appears at least twice among the vertices adjacent to u or w . Since v must receive a color distinct from the colors of u and w , there are at most $\Delta - 1 + 2 = \Delta + 1$ forbidden colors. As $|L(v)| \geq \Delta + 2$, there remains at least one color in $L(v)$ that can be assigned to v . We obtain a linear L -coloring of G , which is a contradiction.

We proved that the 2-degenerate graph G does not contain any ≤ 2 -vertices. The contradiction completes the proof. \square

Since outerplanar graphs are 2-degenerate, we obtain the following corollary:

Corollary 1. *If G is outerplanar, then $\Lambda^l(G) \leq \Delta + 2$.*

3. Graphs with small maximum degree

3.1. Subcubic graphs

As seen in the previous section, the graph $K_{3,3}$ is not linearly 4-colorable. Let G be a graph with maximum degree three, containing at least one ≤ 2 -vertex. Then G is 2-degenerate and we have $\Lambda^l(G) \leq 5$ by Proposition 4. So the

hardest part is to prove that 3-regular graphs have linear list-chromatic number at most five. To show this, we prove a slightly stronger statement:

Theorem 1. *Let G be a graph with maximum degree $\Delta \leq 3$, and L be an assignment of lists of size at least five to the vertices of G . Then there exists a linear L -coloring of G such that the two neighbors of any 2-vertex have distinct colors.*

Proof. Let G be a counterexample of minimum order. There exists an assignment L of lists (with $|L(v)| \geq 5$ for all v) such that there exists no linear L -coloring of G with the property that the two neighbors of any 2-vertex have distinct colors. We can assume that G is connected, otherwise one of the connected components would be a smaller counterexample to the theorem. If G contains a 1-vertex v adjacent to a vertex u , then by the minimality of G , the graph $G \setminus v$ has a linear L -coloring c such that the neighbors of any 2-vertex have distinct colors. By coloring v with a color distinct from $c(u)$ and from the colors of the neighbors of u , we obtain a linear L -coloring of G such that the neighbors of any 2-vertex have distinct colors, which is a contradiction.

If G contains a 2-vertex v with neighbors u and w , let H be the graph obtained from G by removing the vertex v and adding an edge uw if it does not already exist. H has maximum degree at most three and is smaller than G , so there exists a linear L -coloring c of H , such that the neighbors of any 2-vertex have distinct colors. We choose for v a color distinct from $c(u)$, $c(w)$, and from the colors appearing twice in the neighborhood of u , or twice in the neighborhood of w . This forbids at most four colors to v , so we obtain a linear L -coloring of G such that the neighbors of any 2-vertex have distinct colors.

Thus, the graph G is 3-regular, and it contains a shortest cycle u_1, \dots, u_k , with $k \geq 3$. For all $1 \leq i \leq k$, we denote by v_i the neighbor of u_i outside the cycle (that is, distinct from u_{i-1} and u_{i+1} , where all values are taken modulo k). Observe that two vertices v_i and v_j could be the same vertex, but that each v_i is distinct from all the vertices u_j , since otherwise the cycle would not be minimal. Let H be the graph obtained from G by removing the vertices u_1, \dots, u_k . By the minimality of G , there exists a linear L -coloring c of H , such that the neighbors of any 2-vertex have distinct colors. In particular, each vertex v_i has degree at most two in H , so its neighbors have distinct colors and the 3-frugality of v_i will be preserved regardless of the color we assign to u_i .

We now color the vertices u_1, \dots, u_k in this order. We choose for u_1 a color distinct from $c(v_1)$ and $c(v_2)$. For any $2 \leq i \leq k-1$, we choose for u_i a color distinct from $c(u_{i-1})$, $c(v_i)$, and $c(v_{i+1})$. For u_k , we choose a color distinct from $c(u_1)$, $c(u_{k-1})$, $c(v_k)$, and $c(v_1)$. By doing so, we prevent any bicolored cycle containing a vertex v_i , and the 3-frugality of every vertex u_i is respected. But at this point, the cycle u_1, \dots, u_k could still be a bicolored cycle. Hence, if $k \geq 4$, we also forbid the color of u_1 to u_3 while we are coloring this vertex (if $k = 3$ the cycle is a triangle and it cannot be properly bicolored). At most four colors are forbidden to each vertex u_i , so we can choose a color $c(u_i) \in L(u_i)$ for any of them, and the coloring obtained is a linear L -coloring of G . Since G is 3-regular, the additional property that the neighbors of any 2-vertex have distinct colors is trivially verified. This contradicts the assumption that G is a counterexample to the theorem. \square

Since the linear list-chromatic number of $K_{3,3}$ is 5, we propose the following conjecture:

Conjecture 1. *If G has maximum degree $\Delta \leq 3$, and is different from $K_{3,3}$, then $\Lambda^l(G) \leq 4$.*

3.2. Graphs with maximum degree 4

According to Proposition 3, we have $\Lambda^l(K_{4,4}) = 6$. Applying the same method of reducible configurations to graphs with maximum degree 4, we obtain the following theorem, which we suspect not to be tight.

Theorem 2. *If G is a graph with maximum degree $\Delta \leq 4$, then $\Lambda^l(G) \leq 9$.*

Proof. Let G be a counterexample of minimum order: there exists an assignment L of lists (with $|L(v)| \geq 9$ for all v) such that G is not linearly L -colorable. Using the same arguments as in the previous proof, we show that G does not contain any ≤ 3 -vertex. So the graph is 4-regular. We now show that G does not contain any 4-vertices.

Let u be a 4-vertex adjacent to the vertices v , w , x , and y . Let G' be the graph with vertex set $V(G) \setminus \{u\}$ and edge set $E(G) \setminus \{uv, uw, ux, uy\} \cup \{vw, xy\}$ (see Fig. 2). Note that the edges vw and xy may already exist in G . Let c be a

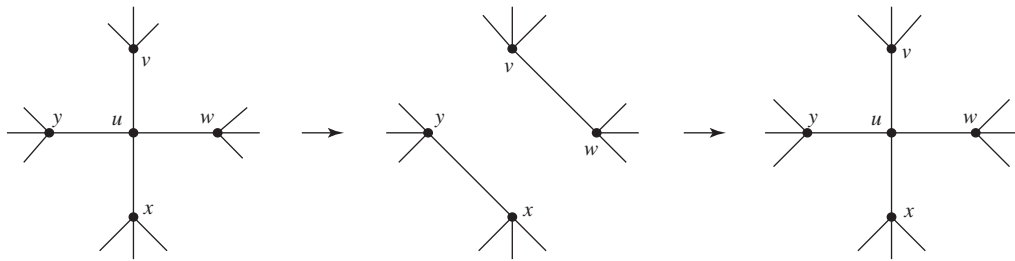


Fig. 2. Elimination of a 4-vertex.

linear L -coloring of G' . We now extend c to the initial graph G : we only have to color the vertex u with a color from its list $L(u)$. We have to choose a color distinct from the colors of v , w , x , and y . The condition of 3-frugality for these four vertices forbids at most four additional colors. If v , w , x , and y have distinct colors, it is impossible to create a bicolored cycle, so we can color u with the ninth color of $L(u)$, and thus obtain a linear L -coloring of G .

Otherwise, we have for example $c(v) = c(y)$ and $c(w) \neq c(x)$. The neighbors of u forbid only three colors, and their 3-frugality forbids at most 4 colors. But it is possible to create a bicolored cycle passing through v and y . To avoid this, we forbid to u the colors of v 's neighbors. This makes only two additional colors, as the third one was already counted to ensure v 's 3-frugality. There are still at most eight forbidden colors for the choice of $c(u)$.

In the last case, we have without loss of generality $c(v) = c(x)$ and $c(w) = c(y)$. The neighbors of u forbid two colors to this vertex. To ensure the 3-frugality of v , w , x , and y we forbid at most four other colors to u . To prevent any bicolored cycle it suffices to forbid to u the colors of v 's and w 's neighbors (six colors, among which two have already been counted). This makes at most eight forbidden colors for the choice of u . So it is possible to color this vertex with a color of its list, and to obtain a linear L -coloring of G . This completes the proof. \square

4. Outerplanar graphs

An outerplanar graph is a graph having a planar representation such that all the vertices are on the external face. In [2], Bonichon et al. proved that any outerplanar graph G could be decomposed into a rooted spanning tree $T(G)$ corresponding to a depth-first search v_1, \dots, v_n in G , and a precise set of edges $M(G)$. Let $f : V(G) \rightarrow V(G)$ be the function defined as follows:

$$f : v_i \mapsto \begin{cases} f(v_{i-1}), & \text{if } v_{i-1} \text{ is } v_i \text{'s father,} \\ v_j \text{ a brother of } v_i \text{ with } j \text{ the maximum index smaller than } i, & \text{otherwise.} \end{cases}$$

The set of edges $M(G)$ is defined as $\{vf(v), v \in V(G)\} \cap E(G)$ (see Fig. 3, where we orient the edges of $M(G)$ from v to $f(v)$ for more clarity). Using this decomposition, we show that a greedy algorithm based on a depth-first search in the spanning tree will enable us to color any outerplanar graph given lists of size at least $\lceil \Delta/2 \rceil + 2$.

Theorem 3. *If G is an outerplanar graph with maximum degree Δ , then $A^l(G) \leq \lceil \Delta/2 \rceil + 2$.*

Proof. Let L be a list assignment for the vertices of G (with $|L(v)| \geq \lceil \Delta/2 \rceil + 2$ for all v). We color the vertices of G during a depth-first search in the spanning tree of the Bonichon et al. decomposition. After $i - 1$ steps we color the vertex v_i . We have to make a distinction between two types of vertices: the v_i whose father is v_{i-1} , and the others.

For the first type of vertices (see Fig. 4), there are two possible situations. If v_{i-1} and v_j are adjacent in G , then they must have different colors and we are certain that no bicolored cycle will be created. We forbid to v_i the colors of v_{i-1} , v_j , plus the colors repeated exactly twice in v_j 's neighborhood. It makes in total $\lceil \Delta/2 \rceil + 1$ forbidden colors: there remains at least one possible color in the list of v_i . If v_{i-1} and v_j are not adjacent while v_{i-1} and v_j have different colors, we are in the same situation as previously. If v_{i-1} and v_j have the same color, a bicolored cycle could be created. To avoid it, we add the color of v_{i-1} in the set of the forbidden colors, whose number stays under $\lceil \Delta/2 \rceil + 1$. Again, there remains at least one possible color in the list of v_i .

For the second type of vertices (see Fig. 5) we also make the distinction between two cases. If v_i is adjacent to $v_j = f(v_i)$ in G , we forbid the colors of v_k , v_j , plus the colors repeated twice in $N(v_k) \setminus v_j$, thus less than $\lceil \Delta/2 \rceil + 1$

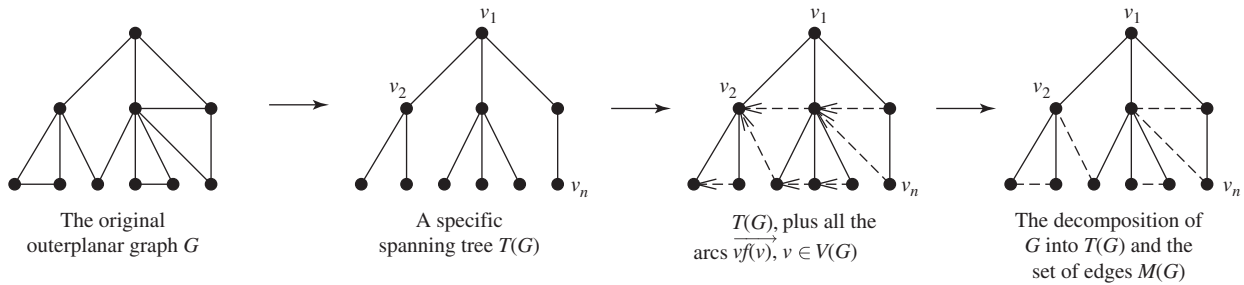


Fig. 3. Decomposition of an outerplanar graph.

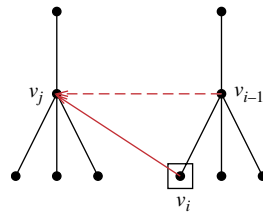


Fig. 4. First type of vertices.

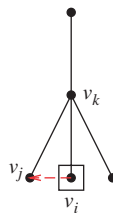


Fig. 5. Second type of vertices.

colors. If v_i is not adjacent to v_j , the forbidden colors are only the color of v_k plus the colors repeated twice in $N(v_k)$, thus less than $\lceil \Delta/2 \rceil$ colors. In each case we will be able to color v_i with a color from its list.

This gives a linear algorithm (the decomposition being linear itself, according to [2]), coloring any outerplanar graph linearly given an assignment of lists of size at least $\lceil \Delta/2 \rceil + 2$. \square

5. Graphs with bounded maximum average degree

Let G be a graph, the maximum average degree of G , denoted by $\text{Mad}(G)$ is defined by

$$\text{Mad}(G) = \max\{2|E(H)|/|V(H)|, H \subseteq G\}.$$

Notice that the maximum average degree of a graph can be computed in polynomial time by using the *Matroid Partitioning Algorithm* due to Edmonds [5,11].

Theorem 4. *Let G be a graph with maximum degree Δ :*

1. *If $\Delta \geq 3$ and $\text{Mad}(G) < \frac{16}{7}$, then $\Lambda^l(G) = \left\lceil \frac{\Delta}{2} \right\rceil + 1$.*
2. *If $\text{Mad}(G) < \frac{5}{2}$, then $\Lambda^l(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil + 2$.*
3. *If $\text{Mad}(G) < \frac{8}{3}$, then $\Lambda^l(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil + 3$.*

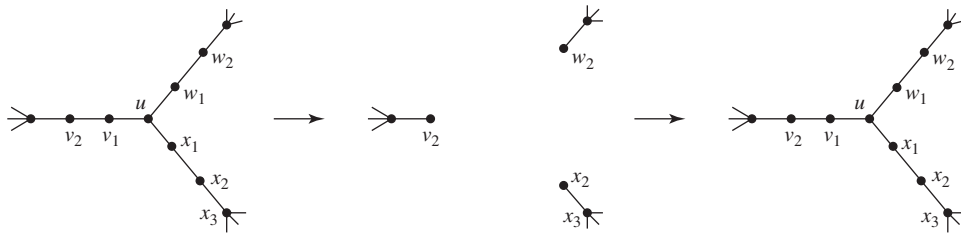


Fig. 6. Elimination of Configuration (C1.3).

Since every planar or projective-planar graph G with girth $g(G)$ verifies $\text{Mad}(G) < 2g(G)/(g(G) - 2)$, we obtain the following corollary:

Corollary 2. *Let G be a planar or projective-planar graph with maximum degree Δ :*

1. *If $\Delta \geq 3$ and $g(G) \geq 16$, then $\Lambda^l(G) = \left\lceil \frac{\Delta}{2} \right\rceil + 1$.*
2. *If $g(G) \geq 10$, then $\Lambda^l(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil + 2$.*
3. *If $g(G) \geq 8$, then $\Lambda^l(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil + 3$.*

Observe that cycles are linearly 3-choosable; hence, we cannot remove the condition on Δ in Theorem 4.1 and Corollary 2.1.

Proof of Theorem 4.1. Let G be a counterexample of minimum order, with $\Delta \geq 3$ and $\text{Mad}(G) < \frac{16}{7}$. There exists an assignment of lists of size at least $\left\lceil \frac{\Delta}{2} \right\rceil + 1$ such that G is not linearly L -colorable. Using the method of reducible configurations, we first prove that G satisfies the following claim:

Claim 1. *G does not contain any of the following configurations:*

- (C1.1). *a 1-vertex,*
- (C1.2). *a 2-vertex adjacent to two 2-vertices,*
- (C1.3). *a 3-vertex adjacent to three 2-vertices, each of them adjacent to a 2-vertex.*

Proof. (C1.1). If G contains a 1-vertex v , let c be a linear L -coloring of $G \setminus v$ (which exists as $G \setminus v$ is a subgraph of G and thus verifies $\text{Mad}(G \setminus v) < \frac{16}{7}$). We now extend c to v : the neighbor u of v forbids one color; we also have to preserve u 's 3-frugality: among its d already colored neighbors ($d \leq \Delta - 1$), there are at worst $\left\lceil \frac{\Delta}{2} \right\rceil - 1$ pairs of vertices having the same color. This forbids at most $\left\lceil \frac{\Delta}{2} \right\rceil$ colors to v . Thus v can be colored with a remaining color in its list $L(v)$, and the coloring obtained is a linear L -coloring of G , which is a contradiction.

(C1.2). If G contains a 2-vertex v adjacent to two 2-vertices u and w , we color the graph $G \setminus v$ linearly with colors belonging to the lists of L (it is possible by the minimality of G). If u and w have distinct colors, we choose for v a color distinct from the colors of its neighbors, and it is impossible to create a bicolored cycle. If u and w have the same color, we forbid it to v , as well as the color of the second neighbor of u . This prevents the creation of any bicolored cycle. There are at most two forbidden colors, what enables us to color v since $\left\lceil \frac{\Delta}{2} \right\rceil + 1 \geq 3$ when $\Delta \geq 3$.

(C1.3). If G contains a 3-vertex adjacent to three 2-vertices, each of them being adjacent to another 2-vertex, then we color the reduced graph H obtained from G by removing the vertices u, v_1, w_1 , and x_1 (see Fig. 6). This reduced graph H is a subgraph of G , and so $\text{Mad}(H) < \frac{16}{7}$. We now have to color the vertices u, v_1, w_1 , and x_1 . For v_1 , we choose a color different from the color of v_2 . For w_1 we take a color different from those of w_2 and v_1 . We color u with a color different from those of v_1 and w_1 . For the last vertex, we have to handle two different cases: if u and x_2 have different colors it is impossible to create any bicolored cycle, so we can take for x_1 a color different from those of u and x_2 . If u and x_2 have the same color, we choose for x_1 a color different from those of x_2 and x_3 (what prevents bicolored cycles coming from x_3). As in the previous situation, there are at most two forbidden colors for each vertex, what enables us to color each of them with a color of its own list. We then obtain a linear L -coloring of G , which is a contradiction. \square

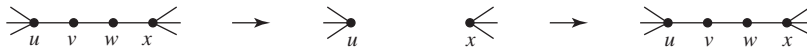


Fig. 7. Elimination of Configuration (C2.2).

We complete the proof of Theorem 4.1 with a discharging procedure. First, we assign to each vertex v a *charge* $\omega(v)$ equal to its degree. We then apply the following discharging rules:

Rule 1. Each ≥ 4 -vertex gives $\frac{2}{7}$ to each adjacent 2-vertex.

Rule 2. Each 3-vertex gives $\frac{2}{7}$ to each adjacent 2-vertex neighbor of another 2-vertex, and $\frac{1}{7}$ to each adjacent 2-vertex which is not neighbor of a 2-vertex.

Let $\omega^*(v)$ be the charge of v after the procedure. Let v be a k -vertex ($k \geq 2$, as G does not contain Configuration (C1.1)).

- If $k = 2$, v receives $\frac{2}{7}$ if it is adjacent to a ≥ 4 -vertex or to a 3-vertex and a 2-vertex. Else v must be adjacent to two 3-vertices (Configuration (C1.2) does not appear in the graph), and will receive two times $\frac{1}{7}$, so $\omega^*(v) \geq 2 + \frac{2}{7} = \frac{16}{7}$.
- If $k = 3$, v gives at most $\frac{2}{7} + \frac{2}{7} + \frac{1}{7}$ (the graph does not contain Configuration (C1.3)), thus $\omega^*(v) \geq 3 - \frac{5}{7} = \frac{16}{7}$.
- If $k \geq 4$, then by Rule 1 $\omega^*(v) \geq k - k \times \frac{2}{7} \geq \frac{20}{7}$.

In any case, $\omega^*(v) \geq \frac{16}{7}$, so $\sum_{v \in V(G)} \omega^*(v) \geq (16n/7)$. Since $\sum_{v \in V(G)} \omega^*(v) = \sum_{v \in V(G)} \omega(v) = \sum_{v \in V(G)} d(v) = 2|E(G)|$, we have

$$\text{Mad}(G) \geq \frac{2|E(G)|}{|V(G)|} = \frac{\sum_{v \in V(G)} \omega^*(v)}{|V(G)|} \geq \frac{16/7|V(G)|}{|V(G)|} = \frac{16}{7}$$

We obtain a contradiction, since $\text{Mad}(G) < \frac{16}{7}$ according to the definition of G .

Proof of Theorem 4.2. Let G be a counterexample of minimum order, with $\text{Mad}(G) < \frac{5}{2}$. There exists an assignment L of lists of size $\lceil \frac{4}{2} \rceil + 2$ such that G is not linearly L -colorable. Using the method of reducible configurations, we first prove that G satisfies the following claim:

Claim 2. G does not contain any of the following configurations:

- (C2.1). a 1-vertex,
- (C2.2). two adjacent 2-vertices,
- (C2.3). a 3-vertex adjacent to three 2-vertices.

Proof. (C2.1). The case of the 1-vertex has already been handled in the previous proof (see Configuration (C1.1)).

(C2.2). If G contains two adjacent 2-vertices v and w , let c be a linear L -coloring of $G \setminus \{v, w\}$ (see Fig. 7). We extend c to the whole graph by finding colors $c(v) \in L(v)$ and $c(w) \in L(w)$ for v and w such that the new coloring c is a linear coloring of G . For v , we choose a color distinct from those of u and x . We also need to preserve u 's 3-frugality; by doing this we forbid at most $\lceil \frac{4}{2} \rceil - 1$ other colors to v . We take for w a color different from those of v and x ; x 's 3-frugality also forbids at most $\lceil \frac{4}{2} \rceil - 1$ other colors to w . At most $\lceil \frac{4}{2} \rceil + 1$ colors are forbidden to v and w , so it is possible to color them with colors from their own lists. We obtain a linear L -coloring of G , which is a contradiction.

(C2.3). If G contains a 3-vertex adjacent to three 2-vertices, let c be a linear L -coloring of the reduced graph H obtained from G by removing the vertices u, x_1 , and w_1 (see Fig. 8). In order to extend c to the whole graph G , we have to find colors for the remaining vertices: w_1, x_1 , and u . We choose for w_1 a color distinct from the colors of w_2 and v_1 , and from then at most $\lceil \frac{4}{2} \rceil - 1$ colors appearing twice in w_2 's neighborhood. We take for u a color different from those of v_1, w_1 , and x_2 . Finally we forbid to x_1 the colors of x_2 and u , as well as most $\lceil \frac{4}{2} \rceil - 1$ colors appearing twice in x_2 's neighborhood. Such a coloring preserves the property of 3-frugality of all the vertices, and since $c(w_1) \neq c(v_1)$

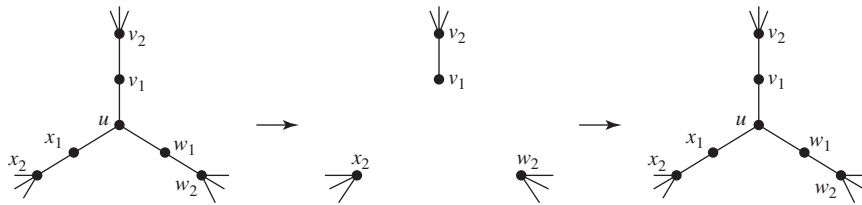


Fig. 8. Elimination of Configuration (C2.3).

and $c(u) \neq c(x_2)$ no bicolored cycle can be created. So we can color each of these vertices with a color from its own list in order to obtain a linear L -coloring of G , which is a contradiction. \square

We complete the proof of Theorem 4.2 with a discharging procedure. First, we assign to each vertex v a charge $\omega(v)$ equal to its degree. We then apply the following discharging rule:

Rule. Each ≥ 3 -vertex gives $\frac{1}{4}$ to each adjacent 2-vertex.

Let $\omega^*(v)$ be the charge of v after the procedure. Let v be a k -vertex of G ($k \geq 2$, as G does not contain Configuration (C2.1)).

- If $k=2$, v is adjacent to two ≥ 3 -vertices (the graph does not contain Configuration (C2.2)), thus $\omega^*(v) \geq 2 + 2 \times \frac{1}{4} = \frac{5}{2}$.
- If $k=3$, v is adjacent to at most two 2-vertices (the graph does not contain Configuration (C2.3)), thus $\omega^*(v) \geq 3 - 2 \times \frac{1}{4} = \frac{5}{2}$.
- If $k \geq 4$, v can be adjacent to k 2-vertices, so $\omega^*(v) \geq k - k \times \frac{1}{4} \geq 3$.

In any case, $\omega^*(v) \geq \frac{5}{2}$, donc $\sum_{v \in V(G)} \omega^*(v) \geq \frac{5n}{2}$. Since $\sum_{v \in V(G)} \omega^*(v) = \sum_{v \in V(G)} \omega(v) = \sum_{v \in V(G)} d(v) = 2|E(G)|$, we have:

$$\text{Mad}(G) \geq \frac{2|E(G)|}{|V(G)|} = \frac{\sum_{v \in V(G)} \omega^*(v)}{|V(G)|} \geq \frac{5/2|V(G)|}{|V(G)|} = \frac{5}{2}$$

We obtain a contradiction, since $\text{Mad}(G) < \frac{5}{2}$ according to the definition of G .

Proof of Theorem 4.3. Let G be a counterexample of minimum order, with $\text{Mad}(G) < \frac{8}{3}$. There exists an assignment L of lists of size $\lceil \frac{4}{2} \rceil + 3$ such that G is not linearly L -colorable. Using the method of reducible configurations, we first show that G satisfied the following claim:

Claim 3. G does not contain any of the following configurations:

- (C3.1). a 1-vertex,
- (C3.2). two adjacent 2-vertices,
- (C3.3). a 3-vertex adjacent to two 2-vertices.

Proof. (C3.1). see Configuration (C1.1).

(C3.2). see Configuration (C2.2).

(C3.3). If G contains a 3-vertex adjacent to two 2-vertices, let c be a linear L -coloring of the reduced graph H obtained from G by removing the vertices u , x_1 , and w_1 (see Fig. 9). This coloring exists, as H is a subgraph of G , and thus $\text{Mad}(H) \leq \text{Mad}(G) < \frac{8}{3}$. We extend c to the whole graph G , by coloring w_1 , x_1 , and u with colors of $L(w_1)$, $L(x_1)$, and $L(u)$, respectively. We take for w_1 a color different from the colors of v and w_2 , and from the $\lceil \frac{4}{2} \rceil - 1$ colors appearing twice in w_2 's neighborhood. We then color u with a color different from those of w_1 , v , x_2 , and from the $\lceil \frac{4}{2} \rceil - 1$ colors appearing twice in v 's neighbors (3-frugality of v). Finally, we color x_1 with a color different from those of u , x_2 , and from at most $\lceil \frac{4}{2} \rceil - 1$ colors among the colors of x_2 's neighbors. So we can color each vertex with a color from its list, and we obtain a linear L -coloring of G , which is a contradiction. \square

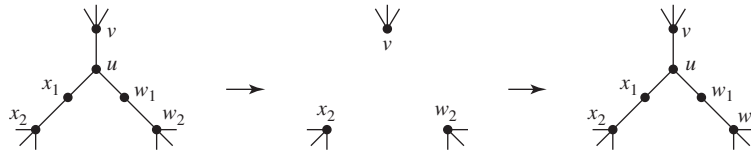


Fig. 9. Elimination of Configuration (C3.3).

We complete the proof of Theorem 4.3 with a discharging procedure. First, we assign to each vertex v a charge $\omega(v)$ equal to its degree. We then apply the following discharging rule:

Rule. Each ≥ 3 -vertex gives $\frac{1}{3}$ to each adjacent 2-vertex.

Let $\omega^*(v)$ be the charge of v after the procedure. Let v be a k -vertex of G ($k \geq 2$, as G does not contain Configuration (C3.1)).

- If $k = 2$, v is adjacent to two ≥ 3 -vertices (G does not contain Configuration (C3.2)), thus $\omega^*(v) \geq 2 + 2 \times \frac{1}{3} = \frac{8}{3}$.
- If $k = 3$, v is adjacent to at most one 2-vertex (G does not contain Configuration (C3.3)), thus $\omega^*(v) \geq 3 - \frac{1}{3} = \frac{8}{3}$.
- If $k \geq 4$, v can be adjacent to k 2-vertices, thus $\omega^*(v) \geq k - k \times \frac{1}{3} \geq \frac{8}{3}$.

In any case, $\omega^*(v) \geq \frac{8}{3}$, so $\sum_{v \in V(G)} \omega^*(v) \geq \frac{8n}{3}$. Since $\sum_{v \in V(G)} \omega^*(v) = \sum_{v \in V(G)} \omega(v) = \sum_{v \in V(G)} d(v) = 2|E(G)|$, we have

$$\text{Mad}(G) \geq \frac{2|E(G)|}{|V(G)|} = \frac{\sum_{v \in V(G)} \omega^*(v)}{|V(G)|} \geq \frac{8/3|V(G)|}{|V(G)|} = \frac{8}{3}.$$

We obtain a contradiction, since $\text{Mad}(G) < \frac{8}{3}$ according to the definition of G .

6. Planar graphs

The square G^2 of a graph G is defined by $V(G^2) = V(G)$, and two vertices are adjacent in G^2 if and only if they are at distance one or two in G . We notice that a proper coloring of G^2 is a linear coloring of G : all the neighbors of a vertex v in G have distinct colors, so the 3-frugality of each vertex is respected, and there are no bicolored paths of length three (and no bicolored cycle, as a consequence).

The best known bound for the chromatic number of the square of a planar graph was obtained by Molloy and Salavatipour (see [10]). They show that if G is a planar graph, then $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 78$ ($\lceil \frac{5}{3}\Delta \rceil + 25$ if $\Delta \geq 241$). The following proposition is a direct consequence of this result:

Observation 1. Let G be a planar graph with maximum degree Δ , then

$$A(G) \leq \begin{cases} \lceil \frac{5}{3}\Delta \rceil + 78, \\ \lceil \frac{5}{3}\Delta \rceil + 25 & \text{if } \Delta \geq 241. \end{cases}$$

Using a structural lemma from Van den Heuvel and McGuinness in [14], we prove the following result, which improves Observation 1.

Theorem 5. If G is a planar graph with maximum degree $\Delta \geq 12$, then $A^l(G) \leq \Delta + 26$.

Proof. Let G be a counterexample with minimum order. There exists an assignment L of lists of size at least $\Delta + 26$ such that G is not linearly L -colorable. In [14], the authors proved the following lemma:

Lemma 1 (Van den Heuvel and McGuinness [14]). Let G be a planar simple graph. Then there exists a vertex v with k neighbors v_1, \dots, v_k with $d(v_1) \leq \dots \leq d(v_k)$ such that one of the following is true:

- (i) $k \leq 2$;

- (ii) $k = 3$ with $d(v_1) \leq 11$;
- (iii) $k = 4$ with $d(v_1) \leq 7$ and $d(v_2) \leq 11$;
- (iv) $k = 5$ with $d(v_1) \leq 6$, $d(v_2) \leq 7$ and $d(v_3) \leq 11$.

Let k, v, v_1, \dots, v_k be as in Lemma 1, and let G' be the graph obtained from G by contracting the edge vv_1 into the vertex v_1 . This graph has maximum degree 12 (case (ii)) or Δ , so by the minimality of G there exists a linear coloring c of G' such that any vertex $u \in V(G')$ is colored with a color $c(u) \in L(u)$. In order to extend c to G , we only need to color v with a color from its list $L(v)$. Choose the color of v different from the colors of v_1, \dots, v_k as well as the colors of the neighbors of v_1, \dots, v_{k-2} if $k \geq 3$. Choose it is also different from the colors appearing twice among the vertices adjacent to v_{k-1} or v_k . In total we forbid at most $5 + 5 + 6 + 10 + (2\Delta - 2)/2 = \Delta + 25$ colors to v . Since $|L(v)| \geq \Delta + 26$, it is possible to find an appropriate color for this vertex.

We now prove that the coloring obtained is linear. Since the coloring c of G' is linear, no color appears more than twice in the neighborhood of v in G . If $k \geq 3$, the colors of the neighbors of v_1, \dots, v_{k-2} are forbidden to v , so the 3-frugality of v_1, \dots, v_{k-2} is preserved and any bicolored cycle passing through v contains v_{k-1} and v_k . The colors appearing twice in $N(v_{k-1})$ or twice in $N(v_k)$ are forbidden, so the 3-frugality of v_{k-1} and v_k is preserved. The colors appearing in $N(v_{k-1})$ and $N(v_k)$ are also forbidden, so v cannot belong to any bicolored cycle. We thus obtain a linear L -coloring of G , which is a contradiction.

7. NP-completeness

Theorem 6. *Deciding whether a bipartite subcubic planar graph is linearly 3-colorable is an NP-complete problem.*

Proof. The proof of the NP-completeness proceeds by a reduction to the problem of 3-coloring of planar graphs, that is an NP-complete problem [7]. Given an instance of this problem—a planar graph H , we need to create a bipartite subcubic planar graph G of a size polynomial in $|V(H)|$ such that G is linearly 3-colorable if and only if H is 3-colorable.

Let \mathcal{M} be the 7×2 grid (see Fig. 10). Observe that in any linear 3-coloring c of \mathcal{M} , we have $c(x_1) = c(x_2)$ and $c(y_1) = c(y_2)$.

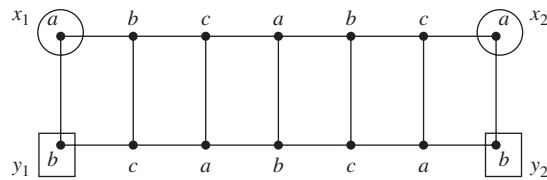


Fig. 10. A linear 3-coloring of the graph \mathcal{M} .

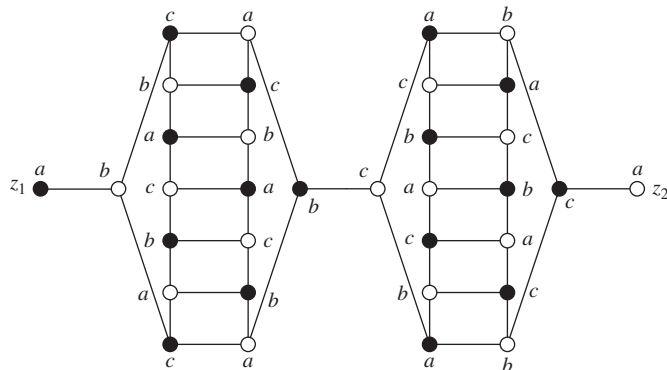


Fig. 11. The graph $\mathcal{N}(z_1, z_2)$. The two stable sets are represented with white and black dots, respectively.

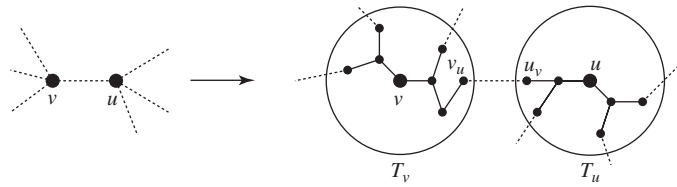


Fig. 12. Transformation of the planar graph into a subcubic bipartite planar graph.

Let $\mathcal{N}(z_1, z_2)$ be the graph depicted in Fig. 11. This graph is bipartite, subcubic, planar, and linearly 3-colorable. Moreover, by the property of \mathcal{M} we have $c(z_1) = c(z_2)$ in any linear 3-coloring c of \mathcal{N} .

To make the reduction, we first replace each d -vertex $u \in V(H)$ by a tree T_u with maximum degree at most 3, having d leaves (each leaf u_v corresponds to a link to a neighbor v of u in H). We then replace each edge xy of these trees by the graph $\mathcal{N}(x, y)$. We then link each vertex u_v to the vertex v_u by an edge (see Fig. 12). Each tree is bipartite, but our construction may not be bipartite at this point: if we color each tree T_u properly with the colors *black* and *white*, two leaves v_w and w_v may be colored with the same color. If this is the case, we subdivide the edge $v_w w_v$, thus creating a new vertex m_{vw} adjacent to v_w and w_v . We then replace the edge $v_w m_{vw}$ by the graph $\mathcal{N}(v_w, m_{vw})$. We repeat this process until the graph obtained is properly 2-colorable, and thus bipartite.

The graph G obtained is planar, bipartite, and subcubic. Each vertex of the tree T_u receives the color of u in the 3-coloring of H . This 3-coloring of the graph G is linear: there is no problem of 3-frugality in the trees, and there are no bicolored cycle (there are no bicolored paths of size at least four in the widgets).

Conversely, in a linear 3-coloring of G , the vertices of a given tree T_u have the same color, which can be used to color u in H . So we easily obtain a 3-coloring of H .

We could have used a 4×2 grid instead of a 7×2 grid to build the widget. All the properties would have been conserved, but the widget would not have been bipartite (it would have contained some C_5). The theorem of NP-completeness would have been a little weaker.

8. Conclusion

An interesting problem would be to find families of planar graphs whose linear chromatic number would be $a\Delta + b$, with $\frac{1}{2} < a \leq 1$ (if such a family exists): we do not know if the bound of Theorem 5 is tight for a certain class of graph. It is also an open problem to know whether $\Lambda^l(G) = \Lambda(G)$ for every graph G .

A generalization of linear coloring can be made, by replacing the condition of 3-frugality by a condition of k -frugality. More precisely, we define the k -forested coloring of a graph G as a proper coloring of the vertices of G such that the union of any two color classes is a forest of maximum degree at most $k - 1$. The k -forested number of a graph G , denoted by $\Lambda_k(G)$, is the smallest number of colors appearing in a k -forested coloring of G .

The lower bound of Proposition 1 can be easily generalized to $\Lambda_k(G) \geq \lceil \frac{\Delta}{k-1} \rceil + 1$ for all graph G of maximum degree Δ . The example described by Yuster [16] can also be generalized in k dimensions in order to prove that $\Lambda_k(G) = \Omega(\Delta^{k/(k-1)})$. However, this construction is less interesting than the construction of Alon et al. [1] for the acyclic chromatic number as soon as $k \geq 5$.

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